where $e = x_m - x_p$, $[a_{ij}] = A_m - A_p$, $[b_{ij}] = B_m - B_p$, α_{ij} and $\gamma_{ij} > 0$, $\beta_{ij}, \rho_{ij}, \delta_{ij}, \sigma_{ij} \geq 0$, $Q = Q^T > 0$, it can be shown that the resulting $\dot{V}$ function is

$$\dot{V} = \overset{①}{e^T(A_m^T Q + QA_m)e} - 2 \overset{②}{\sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \sum_{k=1}^{n} e_k q_{ki} x_{p_j}} \overset{③}{- 2 \sum_{i=1}^{n} \sum_{j=1}^{r} \delta_{ij} \sum_{k=1}^{n} e_k q_{ki} u_j} \tag{12}$$

if the adaptive control gain rates are selected as

$$\dot{a}_{ij} = \alpha_{ij} Z_{ij} + \beta_{ij} \frac{d}{dt} [Z_{ij}] + \rho_{ij} \frac{d^2}{dt^2} [Z_{ij}] \tag{13}$$

$$\dot{b}_{ij} = \gamma_{ij} Y_{ij} + \delta_{ij} \frac{d}{dt} [Y_{ij} / + \sigma_{ij} \frac{d^2}{dt^2} [Y_{ij}] \tag{14}$$

with

$$Z_{ij} = \sum_{k=1}^{n} e_k q_{ki} x_{p_j}, \quad Y_{ij} = \sum_{k=1}^{n} e_k q_{ki} u_j \tag{15}$$

If $\rho_{ij} = \sigma_{ij} = 0$, then Eq. (11) reduces to the V function in Ref. 3, but $\dot{V}$ in Eq. (12) still results from using the reduced version of Eqs. (13) and (14). Note that the last two terms in Eq. (13) are negative definite in e. It is well known that, if A_m is a stable matrix, there exists a p.d. symmetric Q matrix satisfying Eq. (1). Hence from Lyapunov Theory, the plant-model system is asymptotically stable in e. Use of Eq. (1) implies a sufficient condition for stability, the information in terms ② and ③ of Eq. (17) are ignored. Under a set of not-too-restrictive conditions, it can be shown[11] that $\dot{V}$ can be written as

$$\dot{V} = e^T We - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \sum_{k=1}^{n} e_k q_{ki} x_{pj} \tag{16}$$

where

$$W = A_m^T Q + QA_m - \Omega \hat{q} \hat{q}^T \tag{17}$$

If W is negative definite (n.d.), then Eq. (12) is n.d., and if the resulting Q is p.d., then the system will be asymptotically stable in e. Under the constraint of the earlier restrictions, Eq. (17) then may be used in place of Eq. (1) for insuring asymptotic stability. Based on the preceding, the following conjecture is advanced.

Conjecture 4. If A_m is an $n \times n$ real matrix with eigenvalues λ_i and $Re[\lambda_i] < 0$, $i = 1, 2, \ldots n$, W is a negative-definite symmetric matrix, Ω is a positive real number, and $\hat{q}$ is the nth column of an $n \times n$ real matrix $Q = (q_1 q_2 \ldots \hat{q}_n)$, then there exists a unique positive-definite symmetric $n \times n$ matrix Q satisfying Eq. (17).

Use of Eq. (17) can make it possible to obtain a wide selection of q_{ij} values in Eq. (2). If only Eq. (1) is used, then the capability of using $\dot{V}$ is ignored. Based on this result plus Conjecture 1, the following conjecture is advanced.

Conjecture 5. If A_m is a constant, stable, $n \times n$ matrix in phase-variable form, W is a p.d. symmetric matrix, Q is a p.d. symmetric $n \times n$ matrix satisfying Eq. (17), and $\hat{q}$ is the nth column of Q, then the roots x_i, $i = 1, 2, \ldots, n$ of Eq. (17) satisfy $Re[x_i] < 0$, $i = 1, 2, \ldots, n$.

Summary

In general, the nonlinear complex nature of MRAS control laws precludes a clear analysis of a system operating under adaptive control. Using a linearization approach, a number of potential propositions resulted that are of interest in their own right in math analysis, as well as providing a means for interpreting certain adaptive control stability and design conditions. This reinforces the utility of the LECE approach in MRAS systems analysis.

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Turbulent Mixing Characteristics for a D_2/HCl Electric Discharge Gasdynamic Laser

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Introduction

VARIOUS types of electric discharge gasdynamic lasers (EDGDL) have been under investigation at the Naval Research Laboratory.[1,2] This concept is as follows: a diatomic species, such as N_2, CO, D_2, etc., and a monatomic diluent mixture are passed through a glow discharge. A substantial fraction of the discharge energy is dissipated in exciting the vibrational mode of the diatomic molecule. The vibrationally excited gas is expanded through a supersonic

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nozzle and mixed with a second species, such as CO_2, CO, CS_2, C_2H_2, N_2O, or HCl. Downstream in the cavity the diatomic transfers its vibrational energy to this latter specie causing gain and, in some cases, demonstrating lasing in various transitions of the secondary molecule.[1,2]

In all of the reported NRL experiments on this concept, the discharge total pressure was about 50-150 Torr. For a conceptual laser based on a mixture of D_2 and HCl, detailed computer modeling using vibrational kinetics had indicated[3] that that NRL experiment could be improved by operating at higher discharge pressures, i.e., 1 atm, and employing a sustained type of discharge. However, this higher-pressure operation necessitated a redesign of the supersonic mixing nozzles. This Note reports on the analysis used to examine the turbulent mixing characteristics for a conceptual D_2/HCl EDGDL.

Taran et al.[4] have shown theoretically and experimentally that injecting the secondary gas at the nozzle throat gives excellent results for the N_2/CO_2 GDL. It has been demonstrated that satisfactory mixing can be obtained with an increase of the freezing efficiency over simple expansion nozzles or premixed flow. Under the present D_2/HCl EDGDL program, a complete turbulent mixing calculation is not justified due to the exploratory nature of the study. Consequently the problem is solved by using stream function transformation with the turbulent transport modeled by the eddy viscosity approximation.

Method of Analysis

Consider a two-dimensional mixing of two parallel streams with axial pressure gradient. The governing equations are the conservation of mass, momentum, energy, and species, and the equation of state. Introducing the modified von Mises transformation, one replaces the y coordinate by a stream function ψ which inherently satisfies the continuity equation. These equations in the transformed coordinates (x, ψ) are then

$$\frac{\partial u}{\partial x} = -\frac{1}{\rho u}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial}{\partial \psi}\left(\rho u \mu \frac{\partial u}{\partial \psi}\right) \tag{1}$$

$$C_p\frac{\partial T}{\partial x} = \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \rho u \mu\left(\frac{\partial u}{\partial \psi}\right)^2 + \rho u \mu \frac{\partial T}{\partial \psi}\sum_i \frac{Le_i}{Pr} C_{pi}\frac{\partial \chi_i}{\partial \psi} + \frac{\partial}{\partial \psi}\left(\frac{C_p}{Pr}\rho u \mu \frac{\partial T}{\partial \psi}\right) \tag{2}$$

$$\frac{\partial \chi_i}{\partial x} = \frac{\partial}{\partial \psi}\left(\frac{Le_i}{Pr}\rho u \mu \frac{\partial \chi_i}{\partial \psi}\right) \tag{3}$$

where u and v are velocity components in the x (axial) and y (perpendicular) directions, respectively; ρ is the density; p is the pressure; T is the temperature; μ is the eddy viscosity; C_p is the specific heat at constant pressure; Le is the Lewis number; Pr is the Prandtl number; χ is the molar concentration; and subscript i indicates that the quantity is for species i.

Equations (1-3) now can be solved with the appropriate boundary conditions. The boundary conditions at the nozzle wall and at the plane of symmetry ($y=0$) require that all of the gradients vanish. In so doing, we have assumed that there is no boundary layer at the nozzle wall and that the perpendicular pressure gradient is neglected. The axial pressure gradient then is determined from the nozzle area ratio by the isentropic relations. The numerical scheme begins at a given location where the solution is known, and the differential equations will provide the solutions sequentially in the x direction.

At an arbitrary x station, the solution will be obtained in the following way. Using the central differences for the ψ derivatives and backward difference for x derivatives, the preceding equations can be written in a block-tridiagonal matrix form,[5] and with the application of quasilinearization, one can obtain a solution by using the block-tridiagonal algorithm. This method, which requires iterations, has been applied successfully to boundary-layer equations with complex chemical reactions.[6]

There are various successful eddy viscosity models, each developed for a different flow situation. For planar, free turbulent mixing in a coflowing stream, we choose Schetz's[7] model in order to account for the variable-density situation through the appropriate definition of the mass flow defect (or excess).

$$\rho\epsilon = 0.0036\,\rho_e u_e \int_0^{y_e} \left|1 - \frac{\rho u}{\rho_e u_e}\right| \mathrm{d}y$$

where $\rho\epsilon$ is the eddy viscosity, and subscript e indicates the edge condition.

Before we will apply this model to our case, it will be verified against a supersonic turbulent wake-flow measurement. Although the two-dimensional turbulent wake has been studied extensively at low speed, information for supersonic turbulent wakes is scarce. Demetriades[8] has made some mean-flow measurements in the supersonic wake of a very slender, two-dimensional body in a continuous open-circuit wind tunnel at Mach 3 with a stagnation pressure of 730 mm Hg absolute and a stagnation temperature of 38°C. Under these conditions, the role assumed at low speeds by the cylinder diameter as a gaging length of the wake flow is assumed by the wake drag thickness. If the body drag coefficient C_D is associated with a lateral dimension δ, then the wake drag per unit span is

$$\frac{1}{2}\rho_e u_e^2 C_D \delta = \int_{-\infty}^{\infty} \rho u (u_e - u)\,\mathrm{d}y = \text{constant}$$

The constant quantity $C_D\,\delta$, called "virtual body thickness," can be obtained from this expression if we assume that the lateral velocity distribution is similar. From the experimental data, the constant $C_D\,\delta$ is found to be 0.00909 cm, and this value is used to nondimensionalize the axial distance. Demetriades[8] observed that the transverse velocity profiles are Gaussian in the similarity coordinates in the latter half of the wake. Our calculations begin at $x/C_D h = 200$ with the measured centerline velocity and the Gaussian distribution. The calculated downstream velocity profiles continue to be Gaussian. In addition, the rate of decay of the centerline velocity defect $W = 1 - u_c/u_e$ is predicted accurately by Schetz's eddy viscosity model for the turbulent transport in this supersonic flow situation, Fig. 1.

Results and Discussion

Having demonstrated by these comparisons with turbulent flow data that Schetz's model is reasonably accurate for the present application, we now can apply it to determine the mixing characteristics of the D_2/HCl EDGDL. A sketch of the nozzle configuration is shown in Fig. 2. The "cold" HCl/Ar mixtures are injected at the sonic point of the nozzle and form a coaxial flow with the vibrationally excited D_2/Ar mixtures. The nozzle throat height is assumed to be 1.0 mm with a nominal A/A^* of 3.3 to provide for a required expansion to $M = 2.5$ at $X = 1.5$ cm. The exhaust from the nozzle enters into a 30 cm constant cross-sectional area cavity.

For this calculation, the laminar coefficient of viscosity was determined, and was found to be 50-100 times smaller than the eddy viscosity. Thus, our assumption of turbulent mixing appears to be valid.

We have modeled a case in which the primary flow contains approximately equal mole fractions of D_2 and Ar at 1-atm pressure and 300 K. The secondary flow consists of HCl/Ar = 0.18/0.82 at 1.02 atm and 300 K. The slight

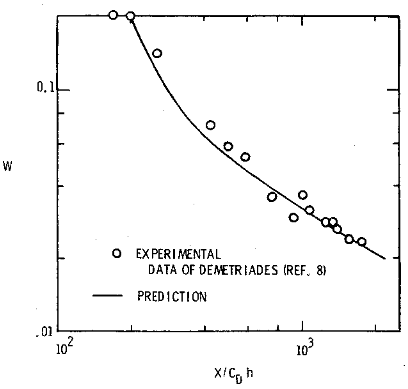

Fig. 1 Comparison of predictions of Schetz's eddy viscosity model with supersonic wake data.

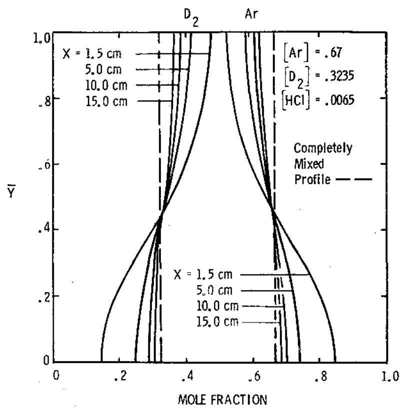

Fig. 4 Concentration profiles for D_2 and Ar at various distances downstream of nozzle throat.

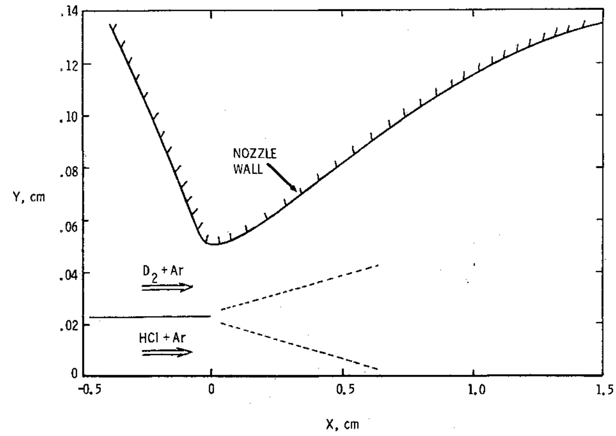

Fig. 2 Supersonic nozzle configuration with parallel injection of HCl/Ar at the throat.

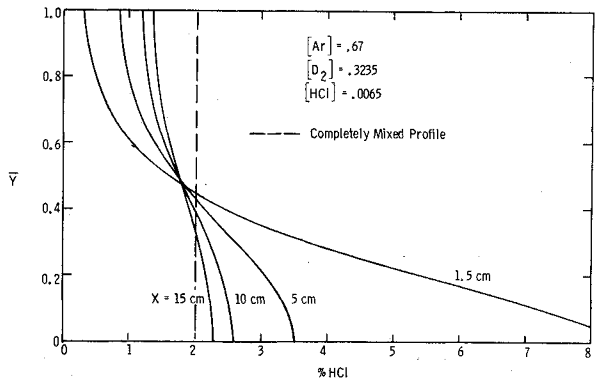

Fig. 5 Concentration profiles for HCl (expressed as percentage HCl relative to D_2) at various locations downstream of the nozzle throat.

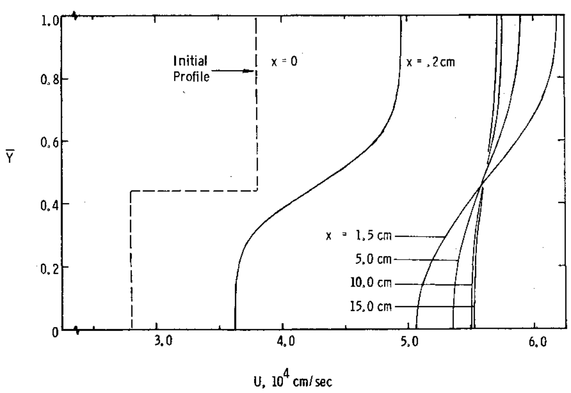

Fig. 3 Velocity profiles at the various axial locations.

overpressure of secondary flow is required to maintain a uniform static pressure at the throat where both flows merge. The calculation starts at the throat where the Mach number is unity. The velocity profiles at various locations in the nozzle and the cavity are given in Fig. 3 as a function of the normalized perpendicular distance, $\bar{y}=y/y_e$.

The concentration profiles for D_2 and Ar are shown in Fig. 4 for various axial distances. It can be seen that the gases are reasonably well mixed at $x \geq 10$ cm. The mixing of the HCl throughout the flow is shown in Fig. 5, where the profiles for HCl, expressed a percentage of HCl relative to D_2 *only,* are given for several downstream stations. Again, it is seen that HCl is reasonably well mixed for $x \geq 10$ cm.

The conclusion of this analysis is that for centerline injection of HCl at the throat of the nozzle, reasonable mixing will occur in a distance of about 10 cm for the gas conditions and mixture compositions specified. Using these results, additional calculations were performed using the full vibrational kinetic model which indicates that high gains and lasing would be expected in the HCl downstream in the cavity. Thus, it would appear from this analysis that a D_2/HCl EDGDL could be designed and operated at a high discharge pressure using a sustained discharge method of excitation.

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